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REMARKS ON THE PRIME VARIETIES

BY

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ABSTRACT

We study the prime varieties of associative algebras over infinite fields of characteristic p. We prove a few properties of the multilinear components of T-prime T-ideals and describe the prime subvarieties of the variety of the algebras satisfying all the identities of the algebra $M_2(F)$ and the identity $x^p = 0$.

Let F be an infinite field of characteristic p, $F\langle X\rangle$ be the free associative algebra over F generated by a countable set X. A T-ideal U of the algebra $F\langle X\rangle$ is called T-prime iff for all T-ideals U_1, U_2 an inclusion $U_1U_2 \subseteq U$ implies one of the inclusions $U_1 \subseteq U$ or $U_2 \subseteq U$. A T-ideal U is called T-semiprime iff for every T-ideal U_1 an inclusion $U_1^n \subseteq U$ implies $U_1 \subseteq U$. A variety of algebras is called prime (semiprime) if its ideal of identities is T-prime (semiprime).

Denote by $N(U)$ the sum of all nilpotent modulo U T-ideals. It is easy to prove $N(U)$ is T-semiprime. Indeed, if it is not so, then there exists a polynomial $g \notin N(U)$ such that $V^2 \subseteq N(U)$, where V is the T-ideal generated by g. Since V^2 is finitely generated as a T-ideal, then $V^2 \subseteq \sum_{i=1}^N U_i$, where the T-ideals U_i are nilpotent modulo U. It follows from this that the ideal V is nilpotent modulo U, i.e. $g \in N(U)$.

Let S be a T-semiprime T-ideal. If the ideal S is not T-prime, then $S = U_1 \cap U_2$ for some T-ideals $U_i \neq S$. Choose a T-ideal S_1 maximal with respect to the property: $S = S_1 \cap U_2$. Then we choose a T-ideal S_2 maximal with respect to the property: $S = S_1 \cap S_2$. The ideals S_i are T-semiprime. Indeed, if, for example, $V^2 \subseteq S_1$ for some T-ideal V, then $(V_1S_2)^2 \subseteq S$, where $V_1 = V + S_1$.

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Since S is T-semiprime, then $V_1 S_2 \subseteq S$ and $S = V_1 \cap S_2$. It follows from this and the definition of S_1 that $V \subseteq S_1$. It is easy to prove, using this remark and transfinite induction, that any T-semiprime T-ideal is equal to the intersection of some family of T-prime T-ideals.

So, the problem of describing the T-prime ideals is very important. In the case of characteristic zero the author ([1]) has proved the following Structure Theorem:

THEOREM: *For any T-ideal U the ideal $N(U)$ is nilpotent modulo U. Any T-semiprime T-ideal is equal to the intersection of some finite number of T-prime T-ideals. A proper T-ideal U is T-prime iff U is an ideal of identities of the Grassman hull of some finite dimensional simple associative superalgebra.*

The problem of describing the prime varieties in the case of characteristic p is open nowadays [2].

In the first section of the paper we divide the T-ideals into two classes — regular and irregular — and prove a few properties of the multilinear components of the regular T-prime T-ideals. In particular, we reduce the study of such T-ideals to the study of the two-sided ideals of the group algebra $FS(n)$(Theorem 1).

The second section is devoted to the prime subvarieties of the variety generated by the algebra of matrices of the second order $M_2(F)$. Recently the author has described the multilinear components of such varieties (unpublished). The proof contains hard straight calculations and is not good to read. In this section we'll give, as an example, part of this description. We'll describe the prime subvarieties of the variety $\mathfrak{V}$ of the algebras satisfying all the identities of $M_2(F)$ and the identity $x^p = 0$.

Let $\widetilde{\mathfrak{P}}_1$ be the variety of all algebras with trace satisfying all the trace identities of $M_2(F)$ and the identities $x^p = 0$ and $\overline{x}^{p-2} = 0$, where $\overline{x} = x - \frac{1}{2}\operatorname{Tr}(x)$. Denote by $\mathfrak{P}_1$ the variety of all the ordinary algebras which satisfy all the ordinary identities of the variety $\widetilde{\mathfrak{P}}_1$. Let $\mathfrak{P}_0$ be the variety of commutative algebras satisfying the identity $x^p = 0$.

The following Theorem is a main result of the section:

THEOREM 2: *If* char $F > 3$, *then $\mathfrak{P}$ is a proper prime subvariety of $\mathfrak{V}$ iff $\mathfrak{P} = \mathfrak{P}_i$ for some $i = 0, 1$.*

The case $p = 2$ is trivial. In this case the variety $\mathfrak{V}$ is prime and has no proper prime subvarieties. In the case $p = 3$ the description is reduced to the open problem: *Is it true that the Grassman algebra of countable rank G satisfies all the identities of the algebra $M_2(F)$?*

1. Some general properties of the prime varieties

Let P be the set of all multilinear polynomials. The set $U \cap P$ is called the multilinear component of the T-ideal U.

An ideal of identities of an algebra A is denoted by $T[A]$.

We start the section with the following lemma:

LEMMA 1: *If U is a proper T-prime T-ideal, then there exists an algebra with unit A such that $T[A] \cap P = U \cap P$.*

Proof: It is sufficient to prove that $f|_{x_j=1} \in U$ for any polynomial $f = f(x_1, \ldots, x_n) \in U \cap P$ and $j \leq n$.

Let $h = h(y_1, \ldots, y_m) \in U$ be a nonzero multilinear polynomial of minimal degree, $y_i \notin \{x_1, \ldots, x_n\}$. The polynomial h can be written in the form

$$h = \sum_{i=1}^{N} w_i y_1 h_i,$$

where the w_i are words and the h_i are polynomials. Without loss of generality we may assume $w_1 = 1, w_i \neq 1$ for $i > 1$. Consider the polynomial

$$g = \sum_{i=1}^{N} f|_{x_j=w_i} z h_i.$$

It is easy to see the polynomial g belongs to the T-ideal, generated by h. Thus $g \in U$. Since $f \in U$ and $w_i \neq 1$ for $i > 1$, then $f|_{x_j=w_i} \in U$ for $i > 1$. It follows from this that $f|_{x_j=1} z h_1 \in U$. Since U is T-prime and $\deg h_1 < \deg h$, then $f|_{x_j=1} \in U$. This proves the lemma. ∎

Now we recall the concepts of algebras with trace and trace identities.

Let A be any associative algebra with unit over the field F, R an associative and commutative algebra with unit over the same field, $C(A)$ the center of A and $\pi \colon R \to C(A)$ a homomorphism of F-algebras. Put $ar = a\pi(r)$ for $a \in A, r \in R$. This turns the algebra A into an R-algebra. Let $\mathrm{Tr}\colon A \to R$ be an arbitrary R-linear mapping satisfying the property: $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ for all $a, b \in A$. We call the set (A, R, π, Tr) an algebra with trace. Sometimes we say simply A is an algebra with trace, having in mind that the algebra R and the mappings π, Tr are given.

Let X be any countable set and $F^\sharp\langle X\rangle$ the free associative algebra with unit generated by the set X. Define an equivalence on the semigroup with unit $\langle X\rangle$ generated by the set X, putting $u_1 \sim u_2$ if and only if there exist elements $v, w \in \langle X\rangle$ such that $u_1 = vw, u_2 = wv$. If $u \in \langle X\rangle$, then we put $\overline{u} = \{v \in \langle X\rangle | v \sim u\}$. Denote by $T\langle X\rangle$ the free associative and commutative algebra with unit generated by all the elements $\mathrm{Tr}(\overline{u})$, where $u \in \langle X\rangle$.

The algebra $\widetilde{F}\langle X\rangle = F^\sharp\langle X\rangle \otimes T\langle X\rangle$ is called the free algebra with trace generated by the set X. If we identify the algebras $F^\sharp\langle X\rangle \otimes 1$ and $F^\sharp\langle X\rangle$, then we have the inclusions:

$$X \subseteq F\langle X\rangle \subseteq F^\sharp\langle X\rangle \subseteq \widetilde{F}\langle X\rangle.$$

We also identify the algebras $T\langle X\rangle$ and $1 \otimes T\langle X\rangle$. Then an arbitrary element $f \in \widetilde{F}\langle X\rangle$ can be written as an F-linear combination of the elements $u_0 \mathrm{Tr}(u_1)\cdots\mathrm{Tr}(u_n)$, where $u_i \in \langle X\rangle, n \geq 0$. We call the elements of this form trace monomials and the elements of the algebra $\widetilde{F}\langle X\rangle$ trace polynomials in the variables from X. We also call the polynomials from $F\langle X\rangle$ ordinary polynomials.

Let A be an algebra with trace, $f = f(x_1, \ldots, x_n) \in \widetilde{F}\langle X\rangle$. We say the algebra A satisfies a trace identity $f = 0$ if for arbitrary $a_1, \ldots, a_n \in A$ the equality $f(a_1, \ldots, a_n) = 0$ is satisfied in A. The ideal

$$\widetilde{T}[A] = \{f \in \widetilde{F}\langle X\rangle | f = 0 \text{ is an identity of } A\}$$

is called the ideal of trace identities of the algebra A. An arbitrary ideal of the algebra $\widetilde{F}\langle X\rangle$ which is an ideal of trace identities of some algebra, is called a $\widetilde{T}$-ideal.

It is obvious that the ideal of trace identities of an arbitrary algebra contains the ideal of the ordinary identities of the same algebra.

A $\widetilde{T}$-ideal Γ is called $\widetilde{T}$-prime iff for every $\widetilde{T}$-ideal Γ_1, Γ_2 an inclusion $\Gamma_1\Gamma_2 \subseteq \Gamma$ implies one of the inclusions $\Gamma_1 \subseteq \Gamma$ or $\Gamma_2 \subseteq \Gamma$.

Let U be a proper T-ideal. In [3] it was proved that $T[M_k(F)] \cap P \subseteq U$ for some k. The minimal number k with this property we call the matrix type of U.

Let k be the matrix type of the T-ideal U. Define the trace in the algebra $M_k(F)$ in the usual way: $\mathrm{Tr}(e_{ij}) = \delta_{ij}$. We call a T-ideal U *regular* iff there exist an ordinary multilinear polynomial $h = h(x_1, \ldots, x_m) \notin U$ such that the algebra $M_k(F)$ satisfies a trace identity of the form

$$g(x_1, \ldots, x_m, x_{m+1}) + \mathrm{Tr}(x_{m+1})h(x_1, \ldots, x_m) = 0 \tag{1}$$

for some ordinary multilinear polynomial g.

The condition of irregularity is quite strong. In the case of characteristic zero, irregular T-prime T-ideals do not exist. Indeed, if U is a T-prime T-ideal, $T[M_k(F)] \subset U$, then, using the well-known results of the structure theory of relatively free PI-algebras, it is easy to see that $T[M_{k-1}(F)] \subseteq U$. This contradicts the definition of the matrix type of U.

In the case of characteristic p the problem of description of the irregular prime T-ideals is very difficult, but for T-ideals whose matrix type equals 2 the problem can be solved using straightforward calculations.

LEMMA 2: *If the matrix type of an irregular T-ideal U equals 2, then*

$$[x, y, z] \in U.$$

Proof: Indeed, define a trace in the algebra $M_2(F)$ in the usual way. Then it is easy to verify that the algebra $M_2(F)$ satisfies the following trace identity:

$$[[x, y] \circ t, z] = \mathrm{Tr}(t)[x, y, z].$$

It follows from this and the definition of the regularity that $[x, y, z] \in U$. This proves the lemma. ∎

Let G be the Grassman algebra of countable rank over the field F. It is well-known that G satisfies the identity

$$[x, y, z] = 0 \tag{2}$$

and that the T-ideal $T[G]$ is T-prime.

It is easy to prove that any multilinear identity, which is not a consequence of (2), implies modulo (2) an identity of the form $[x_1, y_1] \cdots [x_n, y_n] = 0$. In particular, it follows from this that the multilinear components of the T-ideals $T[G]$ and $\{[x, y, z]\}^T$ are equal ($\{g\}^T$ is the T-ideal generated by g).

If U is a T-prime T-ideal, $[x, y, z] \in U$, then applying the remarks formulated above we obtain that either $U \cap P = T[G] \cap P$ or $U \cap P = U_0 \cap P$ ($U_0 = \{[x, y]\}^T$).

So, the problem of describing the multilinear components of the irregular T-prime T-ideals, whose matrix type equals 2, is reduced now to the following problem: *Is it true that the algebra G satisfies all the identities of the algebra $M_2(F)$?*

If $p > 3$, then this problem has a negative solution, because the standard identity of fourth degree does not follow from (2). In the case $p = 3$ the problem is open. We formulate a more general problem:

PROBLEM: *Is it true that the Grassman algebra of a countable rank over an infinite field F of characteristic $p \geq 3$ satisfies all the identities of the algebra $M_{\frac{p+1}{2}}(F)$?*

We note the algebra G satisfies all the multilinear identities of the algebra $M_p(F)$. This follows from the proof of the main theorem in [3].

Now we start to study the multilinear components of the T-prime T-ideals. Define the trace in the algebra $M_k(F)$ in the usual way.

LEMMA 3: *For any proper regular T-prime T-ideal U there exists a $\widetilde{T}$-prime $\widetilde{T}$-ideal Γ such that $U \cap P = \Gamma \cap P$ and $\widetilde{T}[M_k(F)] \subseteq \Gamma$, where k is the matrix type of U.*

Proof: Consider a $\widetilde{T}$-ideal I generated by U:

$$I = UT\langle X\rangle + \mathrm{Tr}(U)\widetilde{F}\langle X\rangle + \widetilde{T}[M_k(F)].$$

First of all we prove the equality $I \cap P = U \cap P$. Indeed, if $u \in I \cap P$, then we have an equality modulo $\widetilde{T}[M_k(F)]$,

$$u = \sum u_i t_i + \sum \mathrm{Tr}(u'_j) w'_j, \tag{3}$$

where $u_i, u'_j \in U; t_i \in T\langle X; w'_j \in \widetilde{F}\langle X\rangle$. Let N be a number such that every element of $T\langle X\rangle$ which appears in the formula (3) can be written as a linear combination of the elements of the form $\mathrm{Tr}(a_1)\cdots\mathrm{Tr}(a_n)$, where $a_i \in \langle X\rangle, n \leq N$. Since U is regular, there exist ordinary multilinear polynomials $h(x_1,\dots,x_m)$ satisfying the properties from the definition of the regularity. Consider the polynomials $h^{(i)} = h(x_1^{(i)},\dots,x_m^{(i)})$, $i = 1,\dots,N$, where the variables $x_j^i \in X$ do not appear in formula (3). The equality (3) implies

$$h^{(1)}\cdots h^{(N)}u = h^{(1)}\cdots h^{(N)}(\sum u_i t_i + \sum \mathrm{Tr}(u'_j)w'_j).$$

Applying the identity (1) to the right side of this equality we obtain the multilinear identity of the algebra $M_k(F)$

$$h^{(1)}\cdots h^{(N)}u = H,$$

where H is an ordinary polynomial, $H \in U$ (the polynomials $h^{(i)}$ "kill" all the traces). Hence we obtain $h^{(1)}\cdots h^{(N)}u \in U$. Since U is prime, $h^i \notin U$, then $u \in U$. Let Γ be a maximal $\widetilde{T}$-ideal, satisfying the properties: $I \subseteq \Gamma, \Gamma \cap P = U \cap P$. Since U is T-prime then Γ is $\widetilde{T}$-prime. This proves the lemma. ∎

Let $\widetilde{P}_n$ be the set of all multilinear polynomials with trace of degree n depending on the variables $x_1, \ldots, x_n$. It follows from the definition of the free algebra with trace that any polynomial $f \in \widetilde{P}_n$ can be written in a unique way as an F-linear combination of monomials

$$u_0(\mathrm{Tr}(1))^l\,\mathrm{Tr}(u_1)\cdots\mathrm{Tr}(u_m), \quad u_i \in \langle X\rangle, \quad m, l \geq 0,$$

which belong to $\widetilde{P}_n$ and satisfy the properties:

1. $u_i \neq 1$ for every $i > 0$;
2. for all $i > 0$ the least number j, such that x_j occurs in u_{i+1}, is greater than the least number k, such that x_k occurs in u_i.

Denote by K the subalgebra with unit of the algebra $\widetilde{F}\langle X\rangle$ generated by the element $\mathrm{Tr}(1)$. Let KS_{n+1} be the group algebra (over K) of the symmetric group of permutations S_{n+1} acting on the set $\{0, 1, \ldots, n\}$. We define a K-linear mapping $\lambda_n : \widetilde{P}_n \to KS_{n+1}$, putting

$$\lambda_n(x_{i_1}\cdots x_{i_s}\,\mathrm{Tr}(x_{j_1}\cdots x_{j_t})\,\mathrm{Tr}(x_{k_1}\cdots x_{k_l})\cdots) = \sigma \in S_{n+1},$$

where σ is a permutation, whose decomposition into the cycles is the following:

$$\sigma = (0, i_1, \ldots, i_s)(j_1, \ldots, j_t)(k_1, \ldots, k_l)\cdots.$$

We see that the symbol 0 plays the role of a label, indicating the non-trace part of the monomial. It follows from the definition of the free algebra with trace that λ_n is an isomorphism of K-modules. If $f \in \widetilde{P}_n, a \in KS_{n+1}$, then we put $fa = \lambda_n^{-1}(\lambda_n(f)a)$, $af = \lambda_n^{-1}(a\lambda_n(f))$.

We call a $\widetilde{T}$-ideal Γ regular if the T-ideal $\Gamma \cap F\langle X\rangle$ is regular.

THEOREM 1: *If Γ is a proper $\widetilde{T}$-prime regular $\widetilde{T}$-ideal, then for every n the set $\lambda_n(\Gamma \cap \widetilde{P}_n)$ is a two-sided ideal of the group algebra KS_{n+1}.*

Proof: Let k be the matrix type of $\Gamma \cap F\langle X\rangle$. Define the trace in the algebra $M_k(F)$ in the usual way. Since Γ is a regular, then for some ordinary multilinear polynomials g and h, $h \notin \Gamma$, the algebra $M_k(F)$ satisfies an identity $u = 0$, where

$$u = g(x_1, \dots, x_{m+1}) + \mathrm{Tr}(x_{m+1})h(x_1, \dots, x_m).$$

By the Proposition from [3] for any permutation $\sigma \in S_{m+2}$ the algebra $M_k(F)$ satisfies the identity $\sigma u = 0$. Let τ_j be the transposition $(0, j)$. If $g = \sum u_i x_{m+1} v_i$, then

$$\tau_{m+1} u = \sum \mathrm{Tr}(x_{m+1} v_i) u_i + h(x_1, \dots, x_m) x_{m+1}.$$

Let $f \in \Gamma \cap \widetilde{P}_n, \sigma \in S_{n+1}$. We prove $\sigma f, f\sigma \in \Gamma \cap \widetilde{P}_n$. Indeed, it is sufficient to prove this inclusion for $\sigma = \tau_j, j = 0, 1, \dots, m+1$. Substituting $x_i = y_i$ into the identities $u = 0$ and $\tau_{m+1} u = 0$, where $y_i \in X \setminus \{x_1, \dots, x_n\}$, we obtain the identities of the algebra $M_k(F)$:

$$\sum u_i' y_{m+1} v_i' + \mathrm{Tr}(y_{m+1}) h(y_1, \dots, y_m) = 0,$$
$$\sum \mathrm{Tr}(y_{m+1} v_i') u_i' + h(y_1, \dots, y_m) y_{m+1} = 0.$$

Using these identities and straightforward calculations, it is not difficult to prove the identities

$$\sum u_i' f|_{x_j = x_j v_i'} = h(y_1, \dots, y_m)(f\tau_j),$$
$$\sum f|_{x_j = u_i' x_j} v_i' = (f\tau_j) h(y_1, \dots, y_m)$$

(it is sufficient to prove the formulas in the case when f is a monomial). Since $f \in \Gamma$, then the left sides of both formulas belong to Γ. It follows from this that $h(y_1, \dots, y_m)(f\tau_j), (f\tau_j)h(y_1, \dots, y_m) \in \Gamma$. Hence we obtain $f\tau_j, f\tau_j \in \Gamma$, because Γ is $\widetilde{T}$-prime and $h \notin \Gamma$. This proves the theorem. ∎

3. Prime subvarieties of $\mathfrak{V}$

In this section $p \neq 2$.

Let A be an algebra with trace, and let L be the set of all elements of A with trace equal to 0. The set L is a Lie subalgebra of Lie algebra $A^{(-)}$. We say the algebra A satisfies a weak identity $f(y_1,\dots,y_n)=0$, where $f\in\widetilde{F}\langle X\rangle$ iff $f(a_1,\dots,a_n)=0$ in A for all $a_i\in L$.

Let Γ be a $\widetilde{T}$-ideal. Denote by Γ' the ideal of weak identities of the algebra $\widetilde{F}\langle X\rangle/\Gamma$. If Γ contains the ideal of trace identities of the algebra $M_2(F)$, then it is obvious that $f(x_1,\dots,x_n)\in\Gamma'$ iff

$$f(\overline{x}_1,\dots,\overline{x}_n)\in\Gamma,$$

where $\overline{x}_i=x_i-\frac{1}{2}\operatorname{Tr}(x_i)$. It is also obvious that Γ is the largest $\widetilde{T}$-ideal contained in Γ'. It follows from this that $\Gamma_1'=\Gamma_2'$ iff $\Gamma_1=\Gamma_2$.

Now we prove a few weak identities of the algebra $M_2(F)$.

First of all $M_2(F)$ satisfies a full linearization of the Cayley–Hamilton identity $X_2(x,y)=0$, where

$$X_2(x,y)=x\circ y-x\operatorname{Tr}(y)-y\operatorname{Tr}(x)+\operatorname{Tr}(x)\operatorname{Tr}(y)-\operatorname{Tr}(xy),$$

which implies the following weak identity:

$$[x\circ y,z]=0. \tag{4}$$

Using this identity we have

$$\begin{aligned}[x,y,z]&=[x,y]z-z[x,y]=2xyz-(x\circ y)z-2zxy+z(x\circ y)\\ &=2xyz+2xzy-2(x\circ z)y=2x(y\circ z)-2y(x\circ z)=2[x,y]z+2(xzy-yzx).\end{aligned}$$

Hence we obtain the following weak identities:

$$[x,y,z]=2x(y\circ z)-2y(x\circ z), \tag{5}$$

$$xzy-yzx=-\frac{1}{2}[x,y]\circ z. \tag{6}$$

Put $\Gamma_2=\widetilde{T}[M_2(F)]$; Γ_0 is the $\widetilde{T}$-ideal generated by the polynomials $[x,y]$ and x^p, Γ_1 is the $\widetilde{T}$-ideal generated by Γ_2 and the polynomials x^p and $\overline{x}^{p-2}$, where $\overline{x}=x-\frac{1}{2}\operatorname{Tr}(x)$.

Lemma 4: *Let Γ be a proper $\widetilde{T}$-prime $\widetilde{T}$-ideal, $\Gamma_2\subseteq\Gamma,\Gamma\cap P\neq\Gamma_0\cap P$. Assume the algebra $A=\widetilde{F}\langle X\rangle/\Gamma$ satisfies a weak identity*

$$h(x_1,\dots,x_{i-1},[x_i,y],x_{i+1},\dots,x_n)=0, \tag{7}$$

for some i, where h is an ordinary polynomial linear with respect to x_i. Then $h \in \Gamma'$.

Proof: Assume $h \notin \Gamma'$. Substituting into (7) $x_i = [y_i, [z,t]]$ and applying to the result the identities (5) and (7), we obtain the following weak identity: $h([z,t] \circ y) = 0$ (remark that $[x_i, y] = 2y_i([z,t] \circ y) - 2[z,t](y_i \circ y)$). Since Γ is $\widetilde{T}$-prime, then it follows from this that the algebra A satisfies a weak identity $[z,t] \circ y = 0$. Then A satisfies the trace identity

$$[z,t] \circ y - [z,t]\operatorname{Tr}(y) = 0. \tag{8}$$

Substituting $t = t^2$ into this identity and applying (8), we obtain

$$\begin{aligned} 0 &= ([z,t,t] + 2t[z,t]) \circ y - ([z,t,t] + 2t[z,t])\operatorname{Tr}(y) \\ &= 2(t[z,t]y + yt[z,t] - t[z,t]\operatorname{Tr}(y)) = 2[y,t][z,t]. \end{aligned}$$

Linearizing the last identity, we get an identity

$$[y,u][z,v] = -[y,v][z,u], \tag{9}$$

which implies an identity

$$[[y,u],[z,v]] = 0. \tag{10}$$

Indeed, using (9), we obtain

$$[y,u][z,v] = -[y,v][z,u] = -[v,y][u,z] = [v,z][y,u] = [z,v][y,u].$$

Substituting into (8) $z = y, t = u, y = [z,v]$ we obtain the identity

$$[y,u] \circ [z,v] = 0.$$

It follows from this and (10) that $[x,u][z,v] \in \Gamma$. Since Γ is $\widetilde{T}$-prime, then $[x,u] \in \Gamma$. Since Γ is proper, then $\Gamma \cap P = \Gamma_0 \cap P$. This proves the lemma. ∎

Let D be any subspace of $\widetilde{F}\langle X \rangle$, and let $f = f(x_1, \ldots, x_n)$ be a multilinear polynomial. We say f is symmetric modulo D if

$$f(x_1, \ldots, x_n) - f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in D$$

for all permutations $\sigma \in S_n$.

LEMMA 5: *Let Γ be a proper $\widetilde{T}$-prime $\widetilde{T}$-ideal, W – $\widetilde{T}$-ideal, $\Gamma_2 \subseteq W \subseteq \Gamma, \Gamma \cap P \neq \Gamma_0 \cap P$. If f is an ordinary multilinear polynomial of minimal degree from the set $\Gamma' \setminus W'$, then f is symmetric modulo W'.*

Proof: Put $g = f(x_1, x_2, \ldots, x_n) - f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. It is sufficient to prove that $g \in W'$, where σ is a cycle of length 2. Without loss of generality we may assume $\sigma = (n-1, n)$. We prove that g can be written modulo W' in the form

$$g = h(x_1, \ldots, x_{n-2}, [x_{n-1}, x_n])$$

for some ordinary multilinear polynomial $h(x_1, \ldots x_{n-1})$. Indeed, using the identity (4), the polynomial g can be written modulo W' as a linear combination of polynomials of the form $ux_{n-1}vx_nw - ux_nvx_{n-1}w$, where v is a word of length ≤ 1. Applying the identity (6) to this polynomial, we obtain that g can be written in the required form. Since $g \in \Gamma'$, then by Lemma 4, $h \in \Gamma'$. It follows from this that $h, g \in W'$, because $\deg h < \deg f$. This proves the lemma. ∎

Now we start to study the $\widetilde{T}$-ideals containing Γ_2 and a polynomial x^p.

LEMMA 6: *Let Γ be a $\widetilde{T}$-ideal, $\Gamma_2 \subseteq \Gamma$. Assume the ideal Γ' contains a polynomial x^n, where $1 < n \leq p, n \neq p-2$. Then $[z, y]x^{n-1} \in \Gamma'$ for all $x, y, z \in X$.*

Proof: Since an identity

$$\sum_{i=0}^{n-1} x^i y x^{n-1-i} = 0 \tag{11}$$

is equivalent to a full linearization of the identity $x^n = 0$, then (11) is an identity modulo Γ'. Using the weak identity $[x^2, y] = 0$, which follows from (4), the left side of (11) is equal modulo Γ' to a polynomial

$$g(y, x) = [(n+1)/2]yx^{n-1} + [n/2]xyx^{n-2}$$

($[r]$ is the integer part of r).

Consider the weak identity $zg(y, x) - yg(z, x) = 0$. The left side of this identity is equal modulo (6) to a polynomial $h(y, x)|_{y=[z,y]}$, where

$$h(y, x) = [(n+1)/2]yx^{n-1} - \tfrac{1}{2}[n/2](x \circ y)x^{n-2}.$$

It follows from this that

$$0 = (\tfrac{1}{2}g + h)|_{y=[z,y]} = \alpha[y, z]x^{n-1},$$

where $\alpha = \frac{1}{2}(3[(n+1)/2]-[n/2])$. It is easy to verify $\alpha \neq 0$ in F if $n \leq p, n \neq p-2$. This proves the lemma. ∎

By applying Lemma 6 twice we get the following corollary:

COROLLARY: *Let Γ be a $\widetilde{T}$-prime $\widetilde{T}$-ideal, $\Gamma_2 \subseteq \Gamma$, $x^p \in \Gamma$. Then $\Gamma_1 \subseteq \Gamma$.*

LEMMA 7: *The $\widetilde{T}$-ideal Γ_1 contains all the polynomials of the form*

$$\overline{x}u_1\overline{x}u_2\cdots\overline{x}u_{p-3}\overline{x},$$

where $\overline{x} = x - \frac{1}{2}\operatorname{Tr}(x), u_i \in \widetilde{F}[X]$.

Proof: It is sufficient to prove the identities $h_i = 0$ modulo Γ, where

$$h_i = \overline{x}u_1\cdots\overline{x}u_i\overline{x}^{p-2-i} = 0$$

for $i = 0, \dots p-2$. We prove these identities by induction on i. The base of induction ($i = 0$) is obvious. Assume $h_i \in \Gamma_1$.

Substituting $x = x + u_{i+1}$ into the identity $h_i\overline{x} = 0$ and, taking a homogeneous part of degree $p-2$ with respect to x, we obtain, using (4), an identity

$$[(p-1-i)/2]h_{i+1} + w = 0,$$

where w is some linear combination of polynomials of the form

$$\overline{x}u_1'\cdots\overline{x}u_i'\overline{x}^{p-2-i},$$

which belong to Γ_1 by the inductive hypothesis. This proves the lemma. ∎

A polynomial $f(x,\dots)$ is called unitary with respect to x if $f(x+1,\dots) = f(x,\dots)$ in $F\langle X\rangle$.

COROLLARY: *If an ordinary polynomial f is homogeneous and unitary with respect to x and $\deg_x f \geq p-2$, then $f \in \Gamma_1$.*

Proof: Indeed, since f is unitary with respect to x, then

$$f(x,\dots) = f(\overline{x},\dots).$$

The left side of this equality belongs to Γ_1 by Lemma 7. This proves the corollary. ∎

LEMMA 8: *Let Γ be a proper $\widetilde{T}$-prime $\widetilde{T}$-ideal, $\Gamma_1 \subseteq \Gamma$. If $\Gamma \neq \Gamma_1$, then $\Gamma_0 \subseteq \Gamma$.*

Proof: Indeed, let $h \in \Gamma' \setminus \Gamma'_1$ be homogeneous with respect to every variable and of minimal degree. Since Γ'_1 contains the Cayley–Hamilton polynomial

$$X_2(x,y) = x \circ y - x\,\mathrm{Tr}(y) - y\,\mathrm{Tr}(x) + \mathrm{Tr}(x)\,\mathrm{Tr}(y) - \mathrm{Tr}(xy)$$

and the polynomial $\mathrm{Tr}(x)$, every polynomial is equal modulo Γ'_1 to some ordinary polynomial. So, we may assume h is ordinary. By the Corollary of Lemma 7 $\deg_x h < p$ for every variable x. Then the identity $h = 0$ is equivalent to its full linearization. It follows from this that a full linearization of the polynomial h does not belong to Γ'_1 and we may assume $h = h(x_1, \ldots, x_n)$ is multilinear. By Lemma 5, h is symmetric modulo Γ'_1.

Put $m = \min(n, p-2)$ and make the following substitution into the polynomial h: $x_i = x_1$ for $i \leq m$. We denote the result of the substitution by g. Since h is symmetric modulo Γ'_1, a multilinear component of the polynomial

$$g|_{x_1 = x_1 + \cdots + x_m}$$

is equal to a polynomial $m!h$ modulo Γ'_1. It follows from this

$$g \notin \Gamma'_1 \tag{12}$$

because $m! \neq 0$ in F.

If $m < p-2$, then $n = m$, $g = \alpha x^n$ for some $\alpha \in F, \alpha \neq 0$. Hence, by Lemma 6, we have an inclusion $[z,y]x^{n-1} \in \Gamma'$, which implies an inclusion $\Gamma_0 \subseteq \Gamma$, because $y^{n-1} \notin \Gamma'$ and Γ is a $\widetilde{T}$-prime $\widetilde{T}$-ideal.

If $m = p-2$, then by Lemma 7, $g \in \Gamma'_1$. This contradicts the condition (12). This proves the lemma. ∎

LEMMA 9: *$\widetilde{T}$-ideal Γ_1 is a $\widetilde{T}$-prime.*

Proof: In the case $p = 3$ the lemma is trivial. Assume $p > 3$.

If the conclusion of the lemma is not true, then by Lemma 8 every $\widetilde{T}$-prime $\widetilde{T}$-ideal containing Γ_1 contains Γ_0. It follows from this that

$$\Gamma_0^n \subseteq \Gamma_1 \tag{13}$$

for some n, because Γ_0 is a finitely generated $\widetilde{T}$-ideal. This can be proven using the methods from the introduction of the paper.

Let Γ be the $\widetilde{T}$-ideal generated by Γ_2 and the polynomial x^p. Applying Lemma 6 and (13), we have an inclusion $\Gamma_0^N \subseteq \Gamma$ for some N. It follows from this that the algebra

$$A = \widetilde{F}\langle X\rangle/\Gamma$$

satisfies an identity of Lie solvability of some degree and the identity $x^p = 0$. Linearizing the identity $x^p = 0$ we get the identity $q = 0$, where

$$q = \sum_{\sigma \in S(p-1)} [x_p, x_{\sigma(1)}, \ldots, x_{\sigma(p-1)}].$$

It is evident that the identity $q = 0$ is equivalent to the Engel identity $[x, y, \ldots, y] = 0$ of degree p. Then by the Theorem of Higgins the algebra A satisfies the identity of Lie nilpotency

$$[x_1, \ldots, x_k] = 0$$

for some k. Since this identity is multilinear, then

$$[x_1, \ldots, x_k] \in W, \tag{14}$$

where W is the $\widetilde{T}$-ideal generated by all the multilinear polynomials from Γ_2 and the polynomial q.

Now it is sufficient to prove that the inclusion (14) is not true for every k. Actually, the proof of this can be deduced from the paper of Razmyslov [4]. The point is that the conclusion of Theorem 5 from [4] is true not only for the polynomial

$$\gamma(\gamma - 1) \cdots (\gamma - p + 1).$$

It is also true for the polynomial $E(\gamma) = \gamma - 2$ (the proof is the same). So, we have the conclusion: If V is the verbal ideal corresponding to the bilinear form f_E, then $q \in V$ and

$$[x_1, \ldots, x_k] \notin V$$

for every k.

It remains to show that V contains all the multilinear polynomials from Γ_2. Since every multilinear trace identity of $M_2(F)$ follows from the Cayley–Hamilton identity of degree 2 ([3]), then it is sufficient to prove the inclusion $X_2(x_1, x_2) \in V$. By the definition of V we need to verify the equalities in F:

$$f_E(X_2(x_1, x_2), u)|_{\gamma=2} = 0$$

for all trace monomials u depending on x_1, x_2. The left sides of these equalities are integers and the equalities are true in the case of characteristic 0 ([4]), so they are true in F. This proves the lemma. ∎

Now we can prove the main theorem:

THEOREM 2: *If* char $F > 3$, *then* $\mathfrak{P}$ *is a proper prime subvariety of* $\mathfrak{V}$ *iff* $\mathfrak{P} = \mathfrak{P}_i$ *for some* $i = 0, 1$.

Proof: Let U_i be the ideal of identities of the variety $\mathfrak{P}_i$, U a proper T-prime T-ideal, $x^p \in U$, $T[M_2(F)] \subseteq U$. By Lemma 3, corollary of Lemma 6 and Lemma 8 we have an equality $U \cap P = \Gamma_i \cap P$ for some $i = 0, 1$. If this equality is true for $i = 0$, then obviously $U = U_0$.

Assume $U \cap P = \Gamma_1 \cap P$, $U \neq U_1$. Let $f \in U \setminus U_1$ be homogeneous with respect to every variable and of minimal degree. If $\deg_x f = n$, then the polynomial f can be written in the form

$$f = \sum_{i=0}^{n} x^k f_k,$$

where the polynomials f_k are homogeneous and unitary with respect to x, $\deg_x f_k = n - k$. Let m be the minimal number for which $x^m f_m \notin U_1$. Since $x^p \in U_1$, then $m < p$.

Put $h_i = [x_{i1}, x_{i2}] \circ [x_{i3}, x_{i4}]$, where $i = 1, \ldots, m, x_{ij} \in X, x_{ij} \neq x_{ks}$ for $(i,j) \neq (k,s)$; the polynomial f does not depend on x_{ij}. Substitute $x = x + h_1 + \cdots + h_m$ into the polynomial f and take the homogeneous component of degree $n - m$ with respect to x and linear with respect to every variable x_{ij}. Since the polynomials h_i are central polynomials of $M_2(F)$, the result of this operation is equal modulo $T[M_2(F)]$ to a polynomial $m! h_1 \cdots h_m f_m$. It follows from this that either $h_1 \in U$ or $m = 0$. If $h_1 \in U$, then $U \cap P \neq \Gamma_1 \cap P$. If $m = 0$, then by the Corollary of Lemma 7, $n = \deg_x f_m < p - 2$ and the identity $f = 0$ is equivalent to its full linearization. Since a full linearization of the polynomial f belongs to U_1, then $f \in U_1$. We have obtained a contradiction. This proves the theorem. ∎

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